Reply by Author to A. Wortman

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WE thank Dr. Wortman[1] for his comments and for presenting his numerical calculations that substantiate the asymptotic results presented in Ref. 2. It should be pointed out that neither Eq. (2) nor Eq. (6) was obtained by curve-fitting. They are asymptotic expressions for small C_M and the reader is referred to Ref. 3 for details.

The predicted linearity of the expression for the Stanton number is verified by Fig. 1 of Ref. 1 and, in fact, extends well beyond the limits set in Ref. 2. The consistency of the sign convention apparently remains subjective.

The change in boundary-layer thickness was only one of the variables used to discuss the results for the heat transfer. Figure 1 in Ref. 2 shows the change in T_{aw} with C_M and a discussion of the role of the adiabatic wall temperature in the behavior of the Stanton number is given on p. 739 of Ref. 2.

References

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Comment on "Mean Velocity Profile of a Thick Turbulent Boundary Layer along a Circular Cylinder"

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THE author has seen with much interest the recent report by Chase[1] and the comments of Bradshaw and Patel[2] on it. Bradshaw and Patel cite the example of the turbulent boundary layer flow with transpiration as an example of the failure of the hypothesis first proposed in Ref. 3. This hypothesis, which essentially implies that the same scaling laws that apply to the sublayer should persist into the region of the law of the wall, has not been properly tested by Bradshaw and Patel which has led them to erroneous conclusions. The purpose of this Comment is to show that the hypothesis of Ref. 3 in fact applies equally well to the case of transpiration also and to offer further comments on some of the other points made by the above authors.

The sublayer equation for transpiration

$$(u/v_*) = (v_*/V_w)[\exp(V_w y/v) - 1] \tag{1}$$

is recast into the following form:

$$(v_*/V_w)\ln[1 + (V_w u/v_*^2)] = yv_*/v \tag{2}$$

The form Eq. (2) is suggested by, for example reference to Eq. (4.9) of Townsend.[4] Logically, the law of the wall is then considered in the form

$$(v_*/V_w)\ln[1 + (V_w u/v_*^2)] = K\ln(yv_*/v) + C \tag{3}$$

A check of Eq. (3) with the form of the bilogarithmic law shows that Eq. (3) has all the properties of the bilogarithmic law. Unlike the bilogarithmic law, Eq. (3) tends to the ordinary logarithmic law for small values of $(V_w u/v_*)$. In a manner similar to the bilogarithmic law and Townsend's[4] Eq. (4.9), it also leaves open the nature of the flow at large suction when the term within the brackets in the left-hand side becomes negative. In view of the well known variation of the von Kármán constant with suction and injection, it is to be expected that K will be a function of the flow through the wall. With $C = 6.0$ and $K = (1/2)$ times the normal value (not unusual), a comparison of the values of (u/v_*) from Eq. (3) and the bilogarithmic law for the example given by Bradshaw and Patel[2] is given in Table 1, which shows that the results from the two formulations match within 4%.

Table 1 Comparison of u/v_* from bilogarithmic law and Eq. (3)

yu_*/v	(u/v_*) bilogarithmic law	(u/v_*) Eq. (3)
100	22.8	22.7
1000	32.4	33.7

In view of this further evidence, it is clearly not a matter of luck that the hypothesis of Ref. 3 works. It is perhaps more fruitful to look for possible basic similarities in the two hypotheses. It does seem that the eddy structure, at least in the region of the law of the wall, cannot be considered in isolation from the effects of the wall and the wall effects appear to be stronger than hitherto assumed.

There is, on the other hand, reason to question the assumption that the eddy length scale is proportional to the distance from the wall in axisymmetric flows. It is difficult to believe, for example that the eddy structure on cylinders of, say, 5 mm and 50 mm diameters should be the same at, say 2 mm from the wall even if the thickness of the turbulent boundary layer is the same in both cases. For example, the velocity induced by one ring vortex on another in the two cases will be different even if the strengths of the two vortices is the same in both cases. For this reason, the search for an alternate formulation for the "mixing length" is justified and the final law of the wall in Ref. 3 can be obtained if the mixing length l is taken in the form

$$l = Ka(r/a)^{1/2}\ln(r/a) \tag{4}$$

The use of the stress variation in deriving Eq. (7) in Ref. 1 seems to have brought the results of Refs. 1 and 3 closer. Experimental justification for the assumption of the stress variation, however, is still lacking but all indirect evidences suggest that the form obtained in the sublayer persists in the region of the law of the wall also.

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Comment on "Comparison of Linear and Riccati Equations Used to Solve Optimal Control Problems"

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IN a recent paper,[1] Tapley and Williamson discussed the application of various linear and Riccati methods to the computational solution of linearized versions of the two-point boundary-value problems (TPBVP's) arising in optimal control theory. Part of their conclusion, for the particular control problem considered, was that the Riccati methods "appear to be very 'unstable' from the standpoint of numerical integration techniques." This conclusion was apparently based on the computational experience that frequently "values for the exponents of the Riccati variables become larger than the computer can handle." The present authors believe this "blow-up" phenomenon is probably not numerical in nature, but rather is an indication of an actual singularity in the Riccati variables. In any event, such singularities are well-known[2,3] to occur sometimes in Riccati methods applied to optimal control problems, and consequently the following comments have relevancy.

The main purpose of this Comment is to point out the existence of known methods for continuing Riccati variables past a singularity. One such method is described briefly below, in order to illuminate this point. It is, of course, true that such methods cannot be based entirely upon step-by-step numerical integration schemes. The method described below is based upon the "recursive equations" (also known as "addition formulas") as discussed, for example, by Denman.[4] These relations are well-known within the field of invariant imbedding, and are essentially the same as the "generalized trigonometric identities" recently used by Allen and Wing[5–7] for the purpose of continuing Riccati variables past singularities. Yet another technique for accomplishing this has recently been discussed by Scott[8] and by Casti, Kalaba, and Scott.[9] Some recently published works[2,3] suggest that none of these methods are well-known to workers in control theory.

For convenience, the following discussion is referred to the approach called the "forward Riccati method" by Tapley and Williamson. The notation and terminology of Ref. 1 will be adopted as nearly as feasible. The primary objective of the Riccati method is, in an obvious extension of the notation of Ref. 1, to find $W(t_f, t_o)$ and $S(t_f, t_o)$, where t_f and t_o are given subject to $t_f > t_o$, and $W(t, t_o)$, $S(t, t_o)$ are $n \times n$ matrix functions determined by the initial-value problem

$$\dot{W} = A_{11}W + A_{12} - W(A_{21}W + A_{22}) \quad (1)$$

$$W(t_o, t_o) = 0 \quad (2)$$

$$\dot{S} = (A_{21}W + A_{22})S \quad (3)$$

$$S(t_o, t_o) = I \quad (4)$$

If $W(t, t_o)$ is finite for $t_o \leqq t \leqq t_f$, this poses no computational difficulty, at least in principle. From the identity $W = \Phi_1\Phi_2^{-1}$, it follows that this condition is equivalent to invertibility of $\Phi_2(t, t_o)$ for $0 \leqq t \leqq t_f$, or, equivalently, to the problem

$$\delta\dot{x} = A_{11}\delta x + A_{12}\delta\lambda \quad (5)$$

$$\delta\dot{\lambda} = A_{21}\delta x + A_{22}\delta\lambda \quad (6)$$

$$\delta x(t_o) = \delta\lambda(\tau) = 0 \quad (7)$$

having only the trivial solution for $0 \leqq \tau \leqq t_f$. For any given nominal trajectory, or even for an optimal trajectory, there appears to be no a priori reason to expect these conditions to hold. Of course, if Eqs. (5–7) have a nontrivial solution for τ near the true value of t_f, then the original nonlinear TPBVP is computationally unstable, which one can possibly expect (or hope) from physical considerations not to be the case. For guessed values of t_f, even this consideration is invalid. It would be of interest to consider the relationship between this possible source of computational instability and the necessity for the under-relaxation scheme for h as described by Tapley and Williamson.

Let $T(t, t_o)$, $P(t, t_o)$, and $Q(t, t_o)$ be the $n \times n$ matrix functions defined by the initial-value system

$$\dot{T} = -T(A_{21}W + A_{22}) \quad (8)$$

$$T(t_o, t_o) = I \quad (9)$$

$$\dot{P} = [A_{11} - WA_{21}]P \quad (10)$$

$$P(t_o, t_o) = I \quad (11)$$

$$\dot{Q} = -TA_{21}P \quad (12)$$

$$Q(t_o, t_o) = 0 \quad (13)$$

Note that $S(t, t_o) = T(t, t_o)^{-1}$. For arbitrary τ, let $W(t, \tau)$ $T(t, \tau)$, $P(t, \tau)$, and $Q(t, \tau)$ be defined by Eqs. (1, 2, and 8–13), with $t_o \to \tau$. If $t_o < t_1 < t_2$, the addition formulas

$$W(t_2, t_o) = W(t_2, t_1 + P(t_2, t_1)[I - W(t_1, t_o)Q(t_2, t_1)]^{-1} \times W(t_1, t_o)T(t_2, t_1) \quad (14)$$

$$T(t_2, t_o) = T(t_1, t_o)[I - Q(t_2, t_1)W(t_1, t_o)]^{-1}T(t_2, t_1) \quad (15)$$

follow from the semigroup property of the fundamental matrix for linear systems. If $t_1 - t_o$ and $t_2 - t_1$ are sufficiently small so that $W(t, t_o)$ and $W(t, t_1)$, respectively, are finite for $t_o \leqq t \leqq t_1$ and $t_1 \leqq t \leqq t_2$, and $I - W(t_1, t_o)Q(t_2, t_1)$ is invertible, then the quantities on the right-hand side of Eqs. (14) and (15) can be obtained by numerical integration of Eqs. (1, 2, and 8–13). With $W(t_2, t_o)$ and $T(t_2, t_o)$ known, these equations can then be integrated from $t = t_2$ to some $t = t_3$ over which W is finite and which furthermore is such that $I - W(t_2, t_o) \cdot Q(t_3, t_2)$ is invertible. The addition formulas then give $W(t_3, t_o)$ and $T(t_3, t_o)$. After enough repetitions, some $t_n = t_f$ is reached, $W(t_f, t_o)$ is known, and $S(t_f, t_o) = T(t_f, t_o)^{-1}$ can be computed.

In order to illustrate further the possibility of continuing Riccati variables across singularities, consider the equation

$$\dot{R} = R + R^2 \sin t \quad (16)$$

which has the solution

$$R(t, t_o) = \frac{2\exp(t - t_o)R(t_o, t_o)}{2 - \exp(t - t_o)[\sin t - \cos t]R(t_o, t_o) + [\sin t_o - \cos t_o]R(t_o, t_o)} \quad (17)$$

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